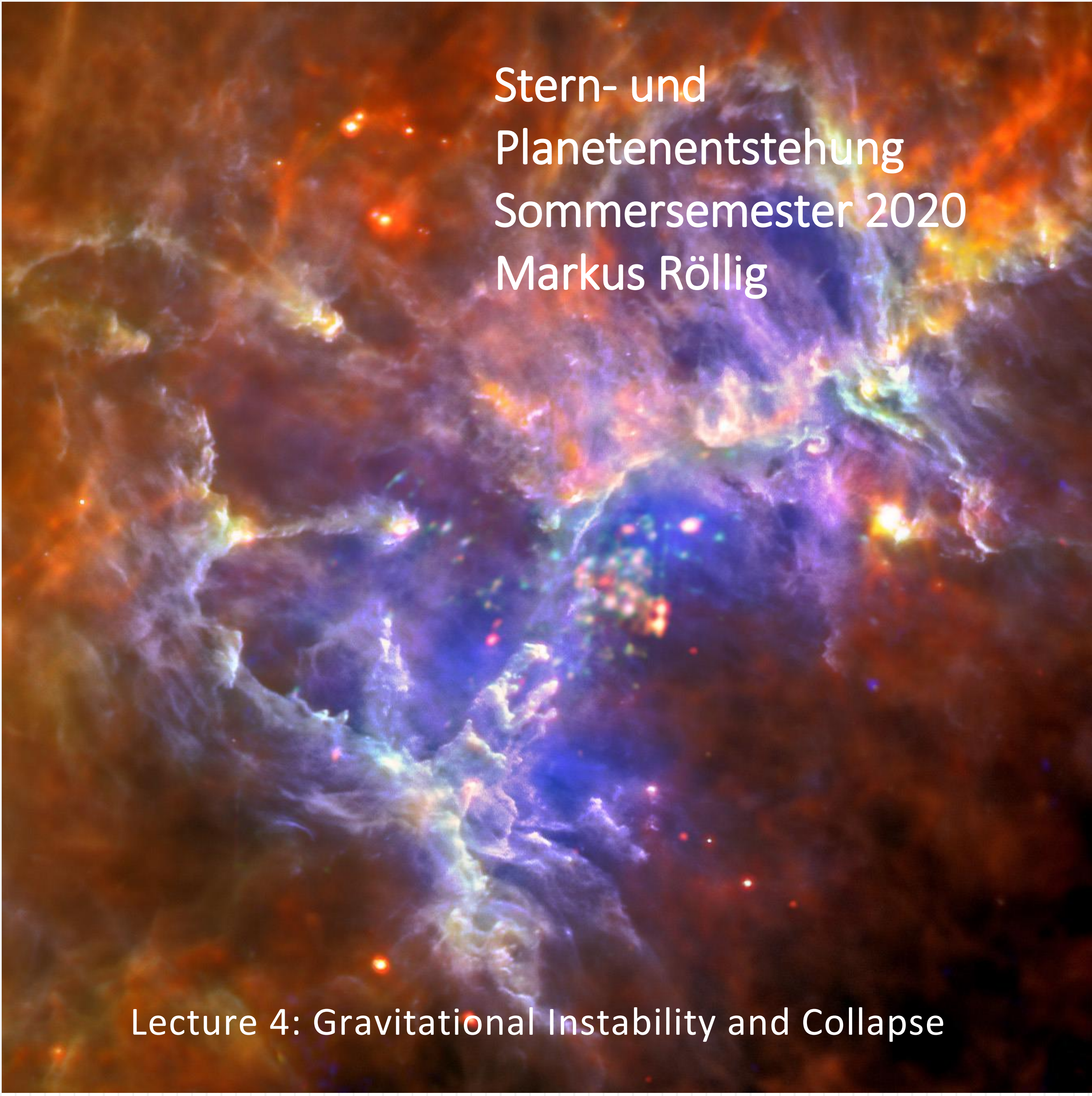


Stern- und Planetenentstehung
Sommersemester 2020
Markus Röllig

Lecture 4: Gravitational Instability and Collapse

http://exp-astro.physik.uni-frankfurt.de/star_formation/index.php

VORLESUNG/LECTURE
Raum: Physik - 02.201a
dienstags, 12:00 - 14:00 Uhr

SPRECHSTUNDE:
Raum: GSC, 1/34, Tel.: 47433, (roellig@ph1.uni-koeln.de)
dienstags: 14:00-16:00 Uhr

Nr.	Thema	Termin
1	Observing the cold ISM	21.04.2020
2	Observing Young Stars	28.04.2020
3	Gas Flows and Turbulence Magnetic Fields and Magnetized Turbulence	05.05.2020
4	Gravitational Instability and Collapse	12.05.2020
5	Stellar Feedback	19.05.2020
6	Giant Molecular Clouds	26.05.2020
7	Star Formation Rate at Galactic Scales	02.06.2020
8	Stellar Clustering	09.06.2020
9	Initial Mass Function – Observations and Theory	16.06.2020
10	Massive Star Formation	23.06.2020
11	Protostellar disks and outflows – observations and theory	30.06.2020
12	Protostar Formation and Evolution	07.07.2020
13	Late Stage stars and disks – planet formation	14.07.2020

4 Gravitational Instability and Collapse

So far we ignored gravity!

4.1 The Virial Theorem

Assume the MHD equations with no viscosity and no resistivity (both unimportant on large scales)

$$\frac{\partial}{\partial t}\rho = -\nabla\cdot(\rho\,\vec{v})$$
$$\frac{\partial}{\partial t}(\rho\vec{v}) = -\nabla\cdot(\rho\,\vec{v}\,\vec{v}) - \nabla P + \frac{1}{4\pi}(\nabla\times\vec{B})\times\vec{B} - \rho\,\nabla\phi$$

here, ϕ is the gravitational potential, so $-\rho\,\nabla\phi$ is the grav. force per unit volume. These are the Eulerian equations (in conservative form).

To simplify them we rewrite them in tensorial form. We define two tensors:

the fluid pressure tensor $\mathbf{\Pi} \equiv \rho\,\vec{v}\,\vec{v} + P\,\boldsymbol{I}$

the Maxwell stress tensor (2nd rank) $\boldsymbol{T}_M \equiv \frac{1}{4\pi}\left(\vec{B}\vec{B} - \frac{B^2}{2}\boldsymbol{I}\right)$

($\boldsymbol{I}$ is the identity tensor). In tensor notation the two are:

$$(\mathbf{\Pi})_{ij} \equiv \rho\,v_i v_j + P\delta_{ij}$$
$$(\boldsymbol{T}_M)_{ij} \equiv \frac{1}{4\pi}\left(B_iB_j - \frac{1}{2}B_kB_k\delta_{ij}\right)$$

Now the momentum equation is:

$$\frac{\partial}{\partial t}(\rho\vec{v}) = -\nabla\cdot(\mathbf{\Pi} - \boldsymbol{T}_M) - \rho\,\nabla\phi$$

because:

$$(\nabla\times\vec{B})\times\vec{B} = \epsilon_{ijk}\epsilon_{jmn}\left(\frac{\partial}{\partial x_m}B_n\right)B_k$$
$$= -\epsilon_{jik}\epsilon_{jmn}\left(\frac{\partial}{\partial x_m}B_n\right)B_k$$

$$= (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn})\left(\frac{\partial}{\partial x_m}B_n\right)B_k$$
$$= B_k\frac{\partial}{\partial x_k}B_i - B_k\frac{\partial}{\partial x_i}B_k$$
$$= \left(B_k\frac{\partial}{\partial x_k}B_i + B_i\frac{\partial}{\partial x_k}B_k\right) - B_k\frac{\partial}{\partial x_i}B_k$$
$$= \frac{\partial}{\partial x_k}(B_iB_k) - \frac{1}{2}\frac{\partial}{\partial x_i}(B_k^2)$$
$$= \nabla\cdot\left(\vec{B}\vec{B} - \frac{B^2}{2}\right)$$

Tensor notation insert:

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \sum_{i=1}^{3}(\vec{a}\times\vec{b})_i\cdot c_i = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}a_jb_kc_i$$
$$(\vec{a}\times\vec{b})_i = \sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}a_jb_kc_i = \epsilon_{ijk}a_jb_kc_i$$
$$\Longrightarrow\quad \vec{a}\times\vec{b} = \epsilon_{ijk}a_jb_k\vec{e}_j = \epsilon_{ijk}a_ib_j\vec{e}_k$$
$$(\vec{a}\times\vec{b})\cdot\vec{c} = \epsilon_{ijk}a_ib_jc_k$$

Example:

$$\epsilon_{123} = \vec{e}_1\cdot(\vec{e}_2\times\vec{e}_3) = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$$

ϵ_{ijk}: Levi-Cita symbol (permutation symbol)

$$\epsilon_{ijk} = \begin{cases}1, & even\ permutation\\ -1, & odd\ permutation\\ 0, & double\ indices\end{cases}\qquad \begin{matrix}\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1\\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1\end{matrix}$$

"Eselsbrücke": 123123 (reading from left +1, from right -1)

Kronecker-delta:

$$\delta_{ij} = \begin{cases}1, & i = j\\ 0, & i \neq j\end{cases}$$

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

$$= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{im}\delta_{jl}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}$$

$$\epsilon_{ijk}\epsilon_{imn} = \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}$$

$$\epsilon_{ijk}\epsilon_{ijk} = 3! = 6$$

Imagine a gas cloud enclosed in some fixed volume *V*. The surface of the cloud is *S*. To specify how the overall distribution of mass inside V changes we write down the moment of inertia:

$$I = \int_V \rho r^2 dV$$

Its change over time is

$$\frac{\partial}{\partial t} I = \dot{I} = \int_V \frac{\partial \rho}{\partial t} r^2 dV$$

$$= -\int_V \nabla \cdot (\rho\, \vec{v}) r^2 dV$$

$$= -\int_V \nabla \cdot (\rho\, \vec{v}\, r^2) dV + 2\int_V \rho\, \vec{v} \cdot \vec{r} dV$$

$$= -\int_S (\rho\, \vec{v}\, r^2) \cdot d\boldsymbol{S} + 2\int_V \rho\, \vec{v} \cdot \vec{r} dV$$

(1st step: V is constant, derivative inside integral, 2nd step: equation of mass conservation, 3rd step: r^2 inside divergence, 4th step: divergence theorem to replace volume with surface integral).

Second time derivative (times ½):

$$\ddot{I} = -\frac{1}{2}\int_S r^2 \frac{\partial}{\partial t}(\rho\, \vec{v}) \cdot d\boldsymbol{S} + \int_V \frac{\partial}{\partial t}(\rho\, \vec{v}) \cdot \vec{r} dV$$

$$= -\frac{1}{2}\frac{d}{dt}\int_S r^2 (\rho\, \vec{v}) \cdot d\boldsymbol{S} - \int_V \vec{r} \cdot [\nabla \cdot (\boldsymbol{\Pi} - \boldsymbol{T}_M) + \rho\, \nabla\phi] dV$$

For any tensor $\boldsymbol{T}$ the following holds:

$$\int_V \vec{r}\cdot\nabla\cdot\boldsymbol{T}\,dV = \int_V x_i \frac{\partial}{\partial x_j} T_{ij} dV$$

$$= \int_V \frac{\partial}{\partial x_j}(x_i T_{ij}) dV - \int_V T_{ij} \frac{\partial}{\partial x_j} x_i dV$$

$$= \int_S x_i T_{ij} dS_j - \int_V \delta_{ij} T_{ij} dV$$

$$= \int_S \vec{r}\cdot\boldsymbol{T}\cdot d\boldsymbol{S} - \int_V \mathrm{Tr}\,\boldsymbol{T}\,dV$$

where $\mathrm{Tr}\,\boldsymbol{T} = T_{ii}$ is the trace of the Tensor $\boldsymbol{T}$.

We note that

$$\mathrm{Tr}\,\boldsymbol{\Pi} = 3P + \rho v^2$$

$$\mathrm{Tr}\,\boldsymbol{T}_M = -\frac{B^2}{8\pi}$$

Inserting this give the virial theorem. Introducing some new terms it reads:

$$\frac{1}{2}\ddot{I} = 2(\mathcal{T} - \mathcal{T}_S) + \mathcal{B} + \mathcal{W} - \frac{1}{2}\frac{d}{dt}\int_S (\rho\,\vec{\mathrm{v}}\,r^2)\cdot d\boldsymbol{S}$$

where:

$$\mathcal{T} = \int_V \left(\frac{1}{2}\rho \mathrm{v}^2 + \frac{3}{2}P\right) dV$$

$$\mathcal{T}_S = \int_S \vec{r}\cdot\boldsymbol{\Pi}\cdot d\boldsymbol{S}$$

$$\mathcal{B} = \frac{1}{8\pi}\int_V B^2 dV + \int_S \vec{r}\cdot\boldsymbol{T}_M\cdot d\boldsymbol{S}$$

$$\mathcal{W} = -\int_V \rho\vec{r}\cdot\nabla\phi dV$$

$\mathcal{T}$: total kinetic energy plus thermal energy of the cloud
$\mathcal{T}_S$: confining pressure on the cloud surface (including thermal pressure and ram pressure of any gas flowing through the surface)
$\mathcal{B}$: difference between the magn. pressure in the cloud interior

(stabilizing) and the magn. pressure plus magn. tension at the cloud surface (trying to collapse).

$\mathcal{W}$: gravitational energy of the cloud (gravitational binding energy + possibly some external grav. field)

$\frac{1}{2}\frac{d}{dt}\int_S(\rho\,\vec{v}\,r^2)\cdot d\boldsymbol{S}$: rate of change of momentum flux across the cloud surface.

$\ddot{I}$ is the integrated form of the acceleration. For a cloud of fixed shape it tells us the rate of change of the cloud's expansion or contraction.

$\ddot{I} < 0$: the terms trying to collapse the cloud are larger
cloud accelerates inward

$\ddot{I} > 0$: the terms that favor expansion are larger
cloud accelerates outward

$\ddot{I} = 0$: cloud is stable

If no gas crosses the cloud surface ($\vec{v} = 0$ at S) and uniform magn. field B_0 at the surface:

$$\boxed{\frac{1}{2}\ddot{I} = 2(\mathcal{T} - \mathcal{T}_S) + \mathcal{B} + \mathcal{W}}$$

with

$$\mathcal{T}_S = \int_S r\,P\,dS$$

$$\mathcal{B} = \frac{1}{8\pi}\int_V (B^2 - B_0)\,dV$$

now $\mathcal{T}_S$ is just the mean radius times pressure at the virial surface and $\mathcal{B}$ just represents the total magn. energy of the cloud minus the magnetic energy of the ambient magnetic background field over the same volume.

In equilibrium ($\ddot{I} = 0$) and if magnetic and surface forces are negligible we have

$$\boxed{2\mathcal{T} = -\mathcal{W}}$$

We define the virial ratio

$$\alpha_{vir} = \frac{2\mathcal{T}}{|\mathcal{W}|}$$

$\alpha_{vir} > 1$ implies $\ddot{I} > 0$

$\alpha_{vir} < 1$ implies $\ddot{I} < 0$

$\alpha_{vir} = 1$ seperates clouds that have enough internal pressure or turbulence to avoid collapse from those that do not.

4.2 Stability Conditions

The virial theorem will help us to understand qualitatively, under what conditions a cloud of gas will be stable against gravitational contraction, and under what conditions it will not be.

$$\frac{1}{2}\ddot{I} = 2(\mathcal{T} - \mathcal{T}_S) + \mathcal{B} + \mathcal{W} - \frac{1}{2}\frac{d}{dt}\int_S (\rho\,\vec{v}\,r^2)\cdot d\boldsymbol{S}$$

Opposing collapse:

- $\mathcal{T}$ (thermal pressure and turbulent motion)
- $\mathcal{B}$ (magnetic pressure and tension)

Favoring collapse:

- $\mathcal{W}$ (self-gravity)
- $\mathcal{T}_S$ (surface pressure)

The surface term (last term on right side) can be positive or negative depending on whether mass is flowing in or out.

4.2.1 Thermal Support and the Jeans Instability

Quick estimate:

Gas pressure always tries to smooth out the gas -> counter collapse

Self-gravity always promotes collapse

expected line between stability and instability: $\alpha_{vir} \approx 1$

isolated, isothermal cloud of Mass M and radius R:

$$\mathcal{T} = \frac{3}{2} M c_s^2$$
$$\mathcal{W} = -a \frac{GM^2}{R}$$

a: depends on internal density structure (of order unity)

$\alpha_{vir} \gtrsim 1$ $\quad\quad M c_s^2 \gtrsim \frac{GM^2}{R}$ $\quad\quad$ => $\quad \boxed{R \gtrsim \frac{G\,M}{c_s^2}}$

or, using the mean density $\rho{\sim}M/R^3$ $\quad\quad R \gtrsim \frac{c_s}{\sqrt{G\rho}}$

Full formal analysis (Jeans (1902))

Consider a uniform, infinite, isothermal medium at rest:

(density: ρ_0, pressure: $P_0 = \rho_0 c_s^2$, velocity: $\vec{v}_0 = 0$)

Equations of HD and self-gravity:

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \vec{v}) = 0$$

conservation of mass

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla P - \rho \nabla \phi$$

conservation of momentum

$$\nabla^2 \phi = 4\pi G \rho$$

Poisson eq. for grav. potential ϕ

$\phi_0, \rho_0, \vec{v}_0, P_0$: exact solutions and $\partial/\partial t$ ->0 if gas is not perturbed

(*Jeans swindle*: There is no function ϕ_0 such that $\nabla^2 \phi_0$ is equal to a non-zero constant value on all space. Approximation to a finite uniform medium)

Perturbation: $\epsilon \ll 1$

$\rho = \rho_0 + \epsilon\rho_1$, $\vec{v} = \epsilon\vec{v}_1$, and $\phi = \phi_0 + \epsilon\phi_1$

We assume a simple, single Fourier mode as perturbation form (simplifies solutions of DEQs):

$$\rho_1 = \rho_a \exp[i(kx - \omega t)] \quad \text{use only } Re(\rho_1)$$

coordinate system choice: wave vector $\vec{k}$ is in the direction of the $\vec{x}$ direction

density perturbation -> which potential perturbation?

$$\nabla^2(\phi_0 + \epsilon\phi_1) = 4\pi G(\rho_0 + \epsilon\rho_1)$$

since $\boxed{\nabla^2\phi_0 = 4\pi G\rho_0}$

$$\nabla^2\epsilon\phi_1 = 4\pi G\epsilon\rho_1$$

$$\nabla^2\phi_1 = 4\pi G\rho_a \exp[i(kx - \omega t)]$$

$$\Rightarrow \boxed{\phi_1 = -\frac{4\pi G\rho_a}{k^2}\exp[i(kx - \omega t)]}$$

$\phi_1 = \phi_a \exp[i(kx - \omega t)]$, therefore:

$$\boxed{\phi_a = -\frac{4\pi G\rho_a}{k^2}}$$

What motion does this induce?

- Insert $\rho = \rho_0 + \epsilon\rho_1$, $\vec{v} = \epsilon\vec{v}_1$, $P = P_0 + \epsilon P_1 = c_s^2(\rho_0 + \epsilon\rho_1)$ and $\phi = \phi_0 + \epsilon\phi_1$ into conservation equations
- linearize them, i.e. expand in powers of ϵ and drop all term of order ϵ^2 and higher, since they become very small

$$\frac{\partial}{\partial t}(\rho_0 + \epsilon\rho_1) + \nabla \cdot [(\rho_0 + \epsilon\rho_1)(\epsilon\vec{v}_1)] = 0$$

$$\cancel{\frac{\partial}{\partial t}\rho_0} + \epsilon\frac{\partial}{\partial t}\rho_1 + \epsilon\nabla \cdot (\rho_0\vec{v}_1) = 0 \qquad \rho_0 = const$$

$$\frac{\partial}{\partial t}\rho_1 + \nabla\cdot(\rho_0\vec{\mathrm{v}}_1) = 0$$

and

$$\frac{\partial}{\partial t}[(\rho_0 + \epsilon\rho_1)(\epsilon\vec{\mathrm{v}}_1)] + \nabla\cdot[(\rho_0 + \epsilon\rho_1)(\epsilon\vec{\mathrm{v}}_1)(\epsilon\vec{\mathrm{v}}_1)]$$
$$= -c_s^2\nabla(\rho_0 + \epsilon\rho_1) - (\rho_0 + \epsilon\rho_1)\nabla(\phi_0 + \epsilon\phi_1)$$
$$\epsilon\frac{\partial}{\partial t}(\rho_0\vec{\mathrm{v}}_1) = -c_s^2\nabla\rho_0 - \rho_0\nabla\phi_0 - \epsilon\left(c_s^2\nabla\rho_1 + \rho_1\nabla\phi_0 + \rho_{0\nabla\phi_1}\right)$$
$$\frac{\partial}{\partial t}(\rho_0\vec{\mathrm{v}}_1) = -c_s^2\nabla\rho_1 - \rho_0\nabla\phi_1$$

$$\boxed{\begin{aligned}\rho_0 &= const\\ \phi_0 &= const\end{aligned}}$$

We assume again: $\vec{\mathrm{v}}_1 = \vec{\mathrm{v}}_a \exp[i(kx - \omega t)]$

$$\frac{\partial}{\partial t}(\rho_a \exp[i(kx - \omega t)]) + \nabla\cdot(\rho_0\vec{\mathrm{v}}_a \exp[i(kx - \omega t)]) = 0$$
$$-i\omega\rho_a \exp[i(kx - \omega t)] + ik\rho_0\mathrm{v}_{a,x}\exp[i(kx - \omega t)] = 0$$
$$-\omega\rho_a + k\rho_0\mathrm{v}_{a,x} = 0$$

$v_{a,x}$: x component of $\vec{\mathrm{v}}_a$

$$\boxed{\mathrm{v}_{a,x} = \frac{\omega\rho_a}{k\rho_0}}$$

$$\frac{\partial}{\partial t}(\rho_0\vec{\mathrm{v}}_a \exp[i(kx - \omega t)])$$
$$= -c_s^2\nabla\rho_a \exp[i(kx - \omega t)] - \rho_0\nabla\phi_a \exp[i(kx - \omega t)]$$
$$-i\omega\rho_0\vec{\mathrm{v}}_a \exp[i(kx - \omega t)]$$
$$= ikc_s^2\rho_a\vec{x}\exp[i(kx - \omega t)] - ik\rho_0\phi_a \exp[i(kx - \omega t)]\,\vec{x}$$
$$\omega\rho_0\mathrm{v}_{a,x} = k(c_s^2\rho_a + \rho_0\phi_a)$$

$$\omega\rho_0\left(\frac{\omega\rho_a}{k\rho_0}\right) = kc_s^2\rho_a - k\rho_0\left(\frac{4\pi G\rho_a}{k^2}\right)$$
$$\boxed{\omega^2 = c_s^2k^2 - 4\pi G\rho_0}$$

Dispersion relation, describing the dispersion of the plane wave solution (relates spatial frequency k to temporal frequency ω).

Implications:

Assume perturbation with short wavelength (large k)

$k \gg 1$ and $c_s^2 k^2 - 4\pi G \rho_0 > 0$

therefore ω is a real number (pos. or neg.)

density: $\rho = \rho_0 + \rho_a \exp[i(kx - \omega t)]$

(uniform background density with small oscillation in space and time on top of it)

since $|\exp[i(kx - \omega t)]| < 1$ everywhere, the oscillation does not grow

Assume perturbation with long wavelength

$k \ll 1$ $c_s^2 k^2 - 4\pi G \rho_0 < 0$

therefore ω is an imaginary number (pos. or neg.)

$\boxed{\exp[-i\omega t]}$ decays to zero (pos. root of ω^2) or grows to infinity (neg. root of ω^2)

At least one solution of the perturbation will not remain small, it will grow: this is an instability.

Arbitrary small perturbations will grow to be large!

Critical size scale beyond which perturbations (only stabilized by pressure) must grow to non-linear amplitude (determined by sign flip of ω):

$$\omega = 0, \quad \Rightarrow \quad \boxed{k_J = \sqrt{\frac{4\pi G \rho_0}{c_s^2}}}$$

The corresponding wavelength is:

$$\boxed{\lambda_J = \frac{2\pi}{k_J}} = \boxed{\sqrt{\frac{\pi c_s^2}{G\rho_0}}}$$

This is known as the Jeans length.

Associated mass scale: Jeans mass $M_J = \rho \lambda_J^3$

$\lambda > \lambda_J$

Typical example: GMC $\rho_0 = 100\ m_p, c_s = 0.2\ \mathrm{km\ s^{-1}}$

$$\boxed{\lambda_J = 3.4\ \mathrm{pc}}$$

⇨ every GMC is larger, and perturbations will always be present

⇨ molecular clouds cannot be stabilized by gas pressure against collapse

How fast does the perturbation grow?

GMC example (size 50 pc): $k = \frac{2\pi}{50\ pc} = \underline{0.12\ \mathrm{pc}^{-1}}$

$$\frac{c_s^2 k^2}{4\pi G \rho_0} = 0.005, \quad and\ \omega \approx \pm i\sqrt{4\pi G \rho_0}$$

Taking the neg. i root (growing mode)

$$\rho_1 \propto \exp\left([4\pi G \rho_0]^{\frac{1}{2}} t\right)$$

⇨ e-folding time for the disturbance to grow is $\underline{\sim 1/\sqrt{G\rho_0}}$

Definition: free-fall time $\boxed{t_{ff} = \sqrt{\frac{3\pi}{32 G \rho_0}}}$ $\sim \frac{1}{\sqrt{G\rho}}$

characteristic time-scale for a medium with negligible pressure-support to collapse.

4.2.2 Magnetic Support and Magnetic Critical Mass

Magnetic terms also opposes collapse.

Consider a uniform spherical cloud of radius R threaded by a magnetic field $\vec{B}$ (uniform inside of the cloud, outside it quickly drops down to the uniform, but much smaller background field $\vec{B}_0$)

Virial theorem:

$$\mathcal{B} = \frac{1}{8\pi}\int_V B^2 dV + \int_S \vec{r}\cdot \boldsymbol{T}_M \cdot d\boldsymbol{S}$$

with

$$\boldsymbol{T}_M \equiv \frac{1}{4\pi}\left(\vec{B}\vec{B} - \frac{B^2}{2}\boldsymbol{I}\right)$$

By assumption, the magnetic pressure inside is dominated by the magn. field inside the cloud:

$$\frac{1}{8\pi}\int_V B^2 dV \approx \frac{B^2 R^3}{6}$$

The surface magn. pressure term:

$$\int_S \vec{r}\cdot \boldsymbol{T}_M \cdot d\boldsymbol{S} = \int_S \frac{B_0^2}{8\pi}\,\vec{r}\cdot d\boldsymbol{S} \approx \frac{B_0^2 R_0^3}{6}$$

The magnetic flux passing through the cloud is $\Phi_B = \pi B R^2$

The same field lines need also to pass through the virial surface (enclosing the cloud at all times) $\Phi_B = \pi B R_0^2$

$$\mathcal{B} \approx \frac{B^2R^3}{6} - \frac{B_0^2 R_0^3}{6} = \frac{1}{6\pi^2}\left(\frac{\Phi_B^2}{R} - \frac{\Phi_B^2}{R_0}\right) \approx \frac{\Phi_B^2}{6\pi^2 R} \qquad R \ll R_0$$

Compare with gravitational term for a uniform cloud of mass M:

$$\mathcal{W} = -\frac{3}{5}\frac{GM^2}{R}$$

$$\boxed{\mathcal{B} + \mathcal{W} = \frac{\Phi_B^2}{6\pi^2 R} - \frac{3}{5}\frac{GM^2}{R} \equiv \frac{3}{5}\frac{G}{R}\left(M_\Phi^2 - M^2\right)}$$

where $$M_\Phi^2 \equiv \sqrt{\frac{5}{2}}\left(\frac{\Phi_B}{3\pi G^{1/2}}\right)$$

M_Φ: magnetic critical mass (const. since $\Phi_B = const$, because of flux freezing)

$M > M\_\Phi$: $\Rightarrow \mathcal{B} + \mathcal{W} < 0$

magnetic force is unable to stop collapse

cloud is called magnetically supercritical

$M < M\_\Phi$: $\Rightarrow \mathcal{B} + \mathcal{W} > 0$

gravity is weaker than magnetism

cloud is called magnetically subcritical

since $\mathcal{B} + \mathcal{W} \propto 1/R$ the resistance grows if the cloud shrinks

A magnetically subcritical cloud will never collapse because magnetism will always stabilize it at a finite radius.

The cloud needs to reduce Φ_B (only possible via e.g. ambipolar diffusion)

For real clouds Tomisaka (1998) gives: $M_\Phi = 0.12\frac{\Phi_B}{G^{1/2}}$ ideal cloud 0.17

Observation of B is difficult, possible for large sample (assuming random orientation)

⇨ observations indicate that magnetic fields in molecular clouds are not strong enough (by themselves) to prevent gravitational collapse)

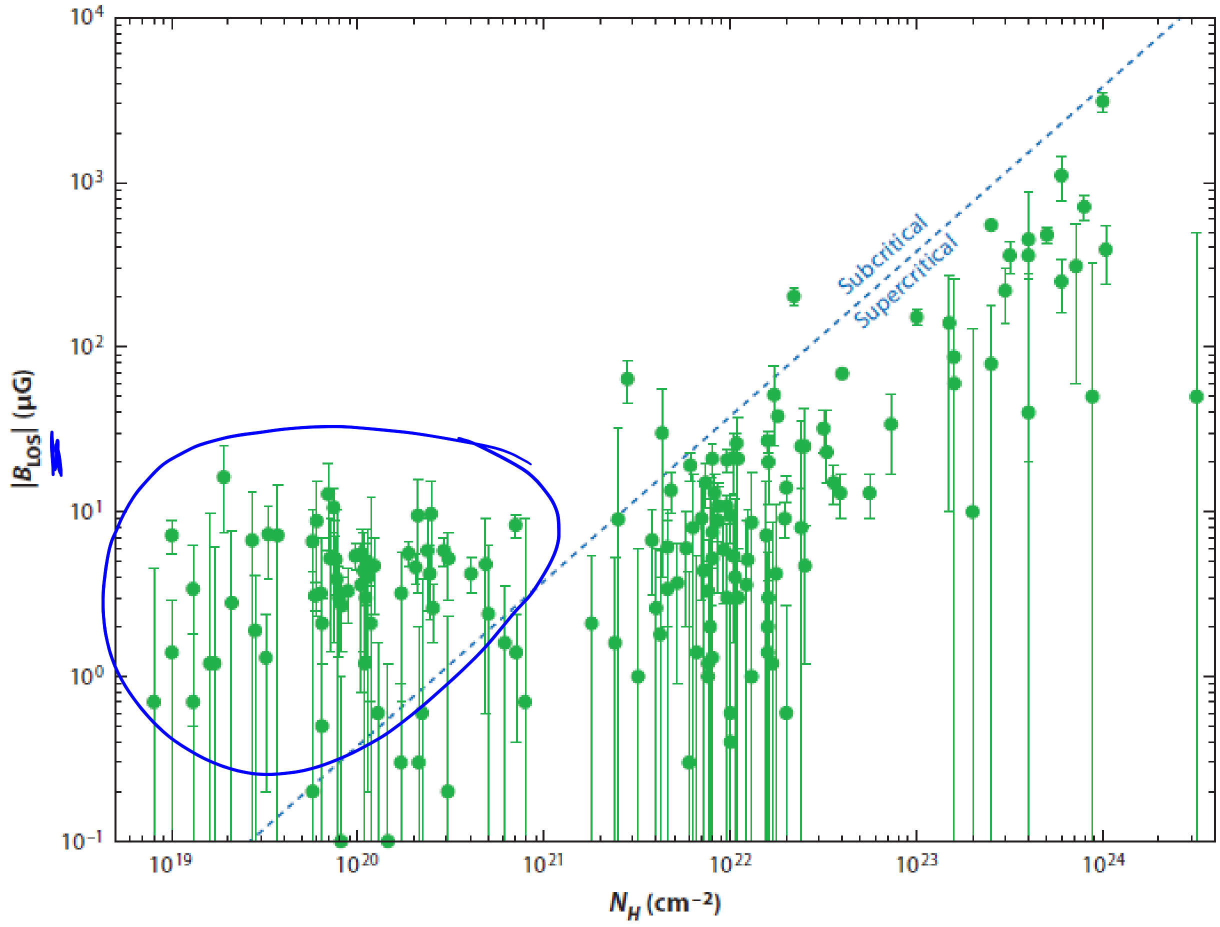


Abbildung 1 Measurements of the line of sight magnetic field strength versus total gas column density (Crutcher 2012)

4.3 Pressureless Collapse

Simplest case of initially-spherically cloud with initial density $\rho(r)$.

Enclosed mass:

$$M_r = \int_0^r 4\pi r'^2 \rho(r') dr'$$

or equivalently:

$$\frac{\partial M_r}{\partial r} = 4\pi r^2 \rho$$

In spherical coordinates:

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \vec{\mathrm{v}}) = 0$$

$$\frac{\partial}{\partial t}\rho + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho \mathrm{v}) = 0$$

v: radial gas velocity

rewrite in terms of mass:

$$\frac{\partial}{\partial t}M_r = 4\pi\int_0^r r'^2\frac{\partial}{\partial t}\rho(r')dr'$$

$$\frac{\partial}{\partial t}\rho = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho\mathrm{v})$$

$$= -4\pi\int_0^r \frac{\partial}{\partial r'}(r'^2\rho\mathrm{v})dr'$$

$$= -4\pi r^2\rho\mathrm{v} = \boxed{-\mathrm{v}\frac{\partial}{\partial r}M_r}$$

Motion of gas: Lagrangian version of momentum equation

$$\rho\frac{D\mathrm{v}}{Dt} = -\frac{\partial}{\partial r}P - \vec{f}_g$$

isothermal $\boxed{P = \rho c_s^2}$

$\vec{f}_g = -GM_r/r^2$

total time derivative of $f(r,t)$

$$\frac{D}{Dt}f(r,t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial r}\dot{r}$$

$$\boxed{\frac{D\mathrm{v}}{Dt}} = \frac{\partial}{\partial t}\mathrm{v} + \mathrm{v}\frac{\partial}{\partial r}\mathrm{v} = \boxed{-\frac{c_s^2}{\rho}\frac{\partial}{\partial r}\rho - \frac{GM_r}{r^2}}$$

Assumption: $c_s = 0$ (pressure proportional to mass and $c_s \sim const.$

gravity grows with *1/R*, soon after collapse starts pressure will become unimportant

⇨ pressureless collapse

$$\boxed{\frac{D\mathrm{v}}{Dt} = -\frac{GM_r}{r^2}}$$

grav. acceleration of a shell is just proportional to all the enclosed mass, which is constant.

$$\mathrm{v} = \dot{r} = -\sqrt{2GM_r}\left(\frac{1}{r_0} - \frac{1}{r}\right)^{1/2}$$

$\boxed{r(t=0) = r_0}$

$r = r_0\cos^2\xi$

$$-2r_0(\cos\xi\sin\xi)\dot{\xi} = -\sqrt{\frac{2GM_r}{r_0}}\left(\frac{1}{\cos^2\xi} - 1\right)^{\frac{1}{2}}$$

$$2(\cos\xi\sin\xi)\dot{\xi} = \sqrt{\frac{2GM_r}{r_0^3}}\tan\xi$$

$$2\cos^2\xi\, d\xi = \sqrt{\frac{2GM_r}{r_0^3}}dt$$

$$\xi + \frac{1}{2}\sin 2\xi = t\sqrt{\frac{2GM_r}{r_0^3}}$$

Collapse complete when $r = 0$, i.e. $\xi = \pi/2$

$$\boxed{t = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM_r}}}$$

If the gas started with uniform density ρ then $M_r = \left(\frac{4}{3}\right)\pi r_0^3\rho$. Then we have:

$$\boxed{t = t_{ff} = \sqrt{\frac{3\pi}{32G\rho}}}$$

This is the free-fall time, required for a uniform sphere of pressureless gas to collapse to infinite density.

Compare with growth time for Jeans instability: $\sim 1/\sqrt{G\rho}$

For a uniform fluid => synchronized collapse (all gas reaches center simultaneously)

Assume $\rho = \rho_c\left(\frac{r}{r_c}\right)^{-\alpha}$ where $(\alpha > 0)$

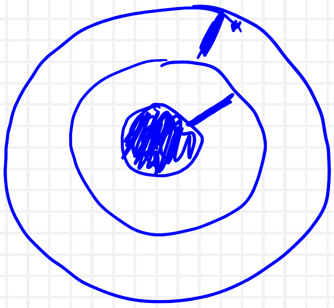
$$M_r = \frac{4}{3-\alpha}\pi\rho_c r_c^3\left(\frac{r}{r_c}\right)^{3-\alpha}$$

then the collapse time is:

$$\boxed{t = \sqrt{\frac{(3-\alpha)\pi}{32G\rho_c}}\left(\frac{r_0}{r_c}\right)^{\alpha/2}}$$

$t \propto r_0^{\alpha/2}$, and > 0 , so the collapse time increases with initial radius r_0

Inside-out Collapse: in centrally concentrated objects inner parts collapse before the outer parts



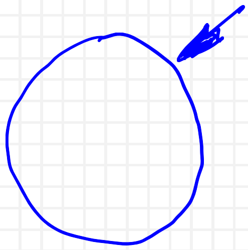
What is the density profile near the center?

For $r \ll r_0$

$$\mathrm{v} \approx \mathrm{v}_{ff} = -\sqrt{\frac{2GM_r}{r}}$$

free-fall velocity v_{ff}: characteristic speed of an object collapsing freely onto a mass *M*.

$$\frac{\partial}{\partial t}M_r = -\mathrm{v}\frac{\partial}{\partial r}M_r = -4\pi r^2 \mathrm{v}\rho$$



near the star ($\mathrm{v} \approx \mathrm{v}_{ff}$), then

$$\rho = \frac{\left(\frac{\partial M_r}{\partial t}\right) r^{-\frac{3}{2}}}{4\pi\sqrt{2GM_r}}$$

For a short time interval $\frac{\partial M_r}{\partial t} \approx const$, then $\rho \propto r^{-3/2}$

We can also estimate the accretion time this implies:

Consider a core of mass $M_c = \left[\frac{4}{3-\alpha}\right]\pi\rho_c r_c^3$. Its last mass element arrives at the center at:

$$t_c = \sqrt{\frac{(3-\alpha)\pi}{32G\rho_c}} = \boxed{\sqrt{\frac{3-\alpha}{3}}t_{ff}(\rho_c)}$$

so the time-averaged accretion rate is

$$\langle \dot{M} \rangle = \sqrt{\frac{3}{(3-\alpha)}\frac{M_c}{t_{ff}(\rho_c)}}$$

Assume the core is a Bonnor-Ebert sphere. Its maximum mass is

$$M_{BE} = 1.18\frac{c_s^4}{\sqrt{G^3 P_s}}$$

P_s: pressure at the surface of the sphere

Assume, that the surface of the core is in thermal pressure balance with its surroundings: $P_S = \rho_c c_S^2$

$$M_{BE} = 1.18\frac{c_s^3}{\sqrt{G^3 \rho_c}}$$

substitute into accretion rate, assume $\sqrt{\frac{3}{(3-\alpha)}} \approx 1$

$$\langle \dot{M} \rangle \approx \frac{\frac{c_s^3}{\sqrt{G^3\rho_c}}}{\frac{1}{\sqrt{G\rho_c}}} = \boxed{\frac{c_s^3}{G}}$$

If we know c_s we can calculate the accretion rate of any object that is marginally stable based on thermal pressure support.

$c_s = 0.19\ km\ s^{-1}$ => $\dot{M} \approx 2 \times 10^{-6} M_\odot yr^{-1}$

For a typical stellar mass of few 0.1 $M_\odot$ we find a **characteristic star formation time** of order 10^5-10^6 yr.